Fractional Mermin-Ho Relation in a Spinor Superconductor

Grayson R. Frazier

Advisor: Yi Li

Johns Hopkins University

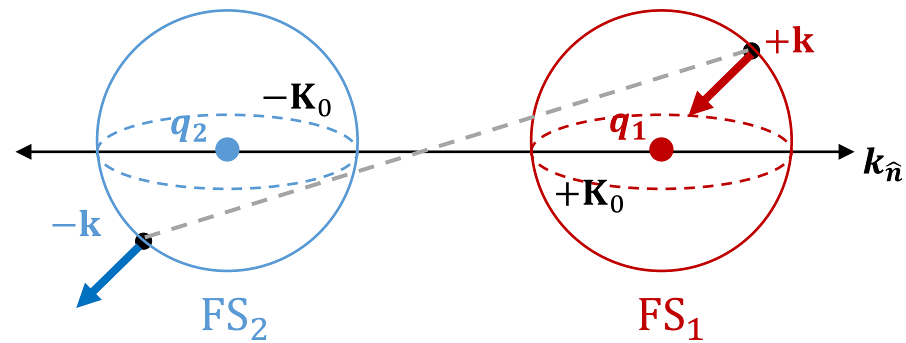
June 19, 2025

arXiv: 2409.09579

Monopole superconductivity

- Can arise in the weak-coupling regime when pairing occurs between two Fermi surfaces with *different* Chern numbers in 3D system

$$|\mathrm{pair}\rangle \sim |\psi_{1,\mathbf{k}}\rangle \otimes |\psi_{2,-\mathbf{k}}\rangle$$



Pair monopole charge: $$q_{\mathrm{pair}} = (1/4\pi) \iint d\mathbf{S}_{\mathbf{k}} \cdot \mathbf{\Omega}_{\mathbf{pair}}(\mathbf{k}) = (\mathcal{C}_1 - \mathcal{C}_2)/2$$

Murakami and Nagaosa, PRL **90**, 057002 (2003)

$q_{\mathrm{pair}} \neq 0 \Rightarrow \mathrm{U}(1)$ phase of the single quasiparticle state and *pairing order* follows a singular representation

- Pairing order is described by monopole harmonics classified by the pair monopole charge

Li and Haldane, PRL **120**, 067003 (2018)

Wu and Yang, Nucl. Phys. B **107**, 365 (1976)

Model of spinor superconductor ($|q_p|$=1/2)

$$H = \sum_{\mathbf{k}} \sum_{\alpha,\beta=\uparrow,\downarrow,} c_\alpha{}^\dagger(k)[\mathcal{H}_c^0(\mathbf{k})]_{\alpha\beta} c_\beta\,(\mathbf{k}) \;+\; \sum_{\mathbf{k}} d^\dagger(k)\mathcal{H}_d^0 d(\boldsymbol{k}) \;+\; \sum_{\mathbf{k},\alpha} \Delta_\alpha(\mathbf{k}) c_\alpha{}^\dagger(\mathbf{k}) d(-\mathbf{k}) + \mathrm{h.c.}$$

Spinful c-fermions
with Weyl SOC
$(\mathcal{C}_c = \pm 1)$

“Spinless” d-fermions
$(\mathcal{C}_d = 0)$

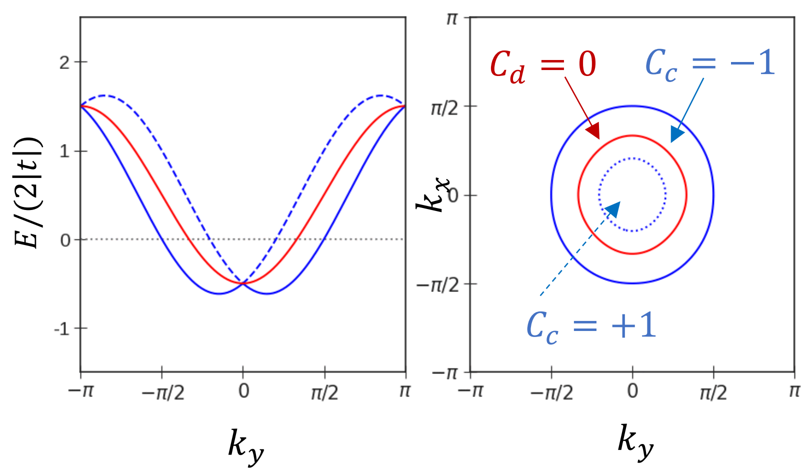
Inter-FS Pairing
$\boldsymbol{\Delta}(\mathbf{k}) = \big(\Delta_\uparrow(\mathbf{k}), \Delta_\downarrow(\mathbf{k})\big)^{\mathbf{T}}$

⇒ Cooper pair acquires nontrivial pair monopole charge $\left|\boldsymbol{q_p}\right| = \mathbf{1/2}$

Tight-binding model:

$$\mathcal{H}_{\mathrm{tbm}} = \sum_{n,n} \Psi_n^\dagger \left(\begin{array}{cc|c} \multicolumn{2}{c|}{[\mathcal{H}_c^0]_{n,n'}} & \begin{array}{c}\Delta_\uparrow \delta_{n,n'} \\ \Delta_\downarrow \delta_{n,n'}\end{array} \\ \hline \Delta_\uparrow^* \delta_{n,n'} & \Delta_\downarrow^* \delta_{n,n'} & -[\mathcal{H}_d^0]_{n,n'} \end{array}\right) \Psi_n$$

$$\Psi_n = \left(c_{n\uparrow}\,, c_{n\downarrow}, d_n^\dagger\right)$$



Fourier transform:

$$\mathcal{H}_{c,2\times 2}(\boldsymbol{k}) = \textstyle\sum_i (2t\cos k_i - \lambda \sin k_i \sigma_i) - \mu_c$$

$$\mathcal{H}_{d,1\times 1}(\boldsymbol{k}) = \sum_i 2t\cos k_i - \mu_d$$

$$\mu_c = \mu_d + \lambda|\sin k_F|$$
$$(m_c = m_d)$$

Effective pairing order parameter

- Decompose into partial wave channels and project to Fermi surface: $\boldsymbol{\Delta}(\mathbf{k}) = \sum_{j,m_j} \boldsymbol{\Delta}_{j,m_j}(\mathbf{k})$

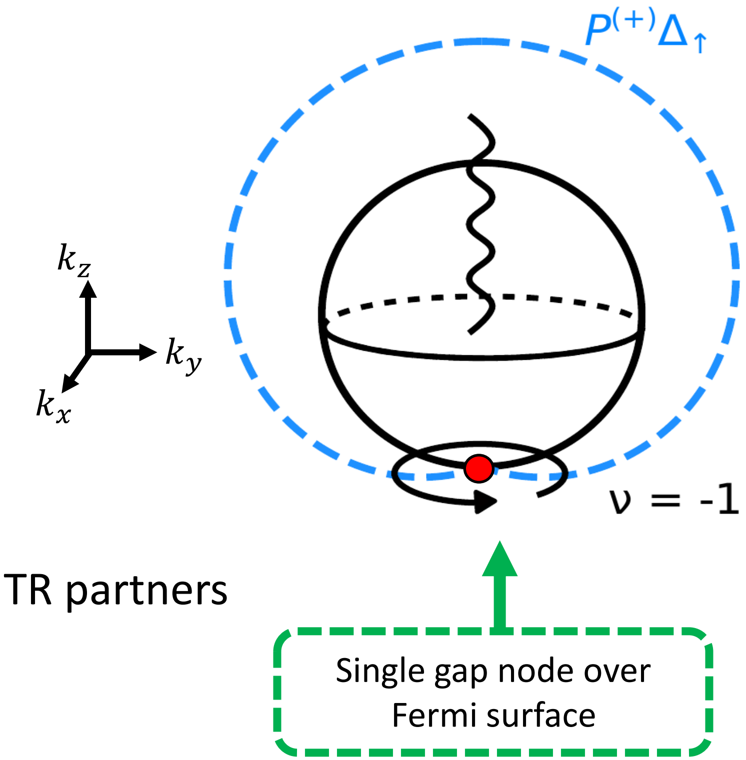
$$P^{(+)}H_\Delta(\mathbf{k})P^{(+)} = \sum_{j,m_j} \boldsymbol{\Delta}_{j,m_j}\mathcal{Y}_{q_{\text{pair}}=-\frac{1}{2};j,m_j}(\hat{\mathbf{k}})\chi_+^\dagger(\mathbf{k})d^\dagger(-\mathbf{k}) + \text{h.c.}$$

- Regardless of pairing mechanism, the pairing order *always* exhibits half-integer charged monopole pairing, following **singular spinor representations**
  - Consider simplest attractive contact interaction,

$$V(\mathbf{k},\mathbf{k}') = V_0\delta_{\mathbf{k},-\mathbf{k}} \Rightarrow \Delta(\mathrm{k}) = (\Delta_\uparrow, \Delta_\downarrow)^{\mathrm{T}}$$

Projected Pairing:

$$\left.\begin{aligned} P^{(+)}\Delta_\uparrow &= \sqrt{2\pi}\Delta_0\mathcal{Y}_{-\frac{1}{2};\frac{1}{2},\frac{1}{2}}(\hat{\mathbf{k}}) = \Delta_0\cos\frac{\theta_k}{2}e_k^{i\varphi_k} \\ P^{(+)}\Delta_\downarrow &= \sqrt{2\pi}\Delta_0\mathcal{Y}_{-\frac{1}{2};\frac{1}{2},-\frac{1}{2}}(\hat{\mathbf{k}}) = \Delta_0\sin\frac{\theta_k}{2}e_k^{-i\varphi_k}\end{aligned}\right\}\ \text{TR partners}$$

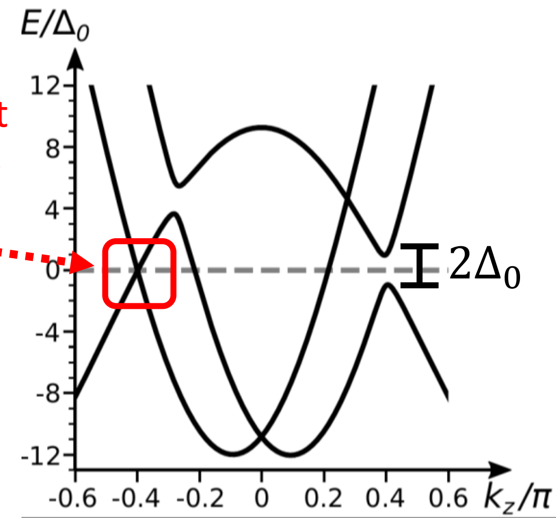


Single emergent BdG Weyl node in lattice model

Low energy effective Hamiltonian:

$$\mathcal{H}(\tilde{\mathbf{k}} - k_F \hat{z}) \approx \begin{pmatrix} -v_1 \tilde{k}_z & v_\parallel(\tilde{k}_x + i\,\tilde{k}_y) \\ v_\parallel(\tilde{k}_x - i\,\tilde{k}_y) & v_2 \tilde{k}_z \end{pmatrix}$$

Single emergent BdG Weyl node

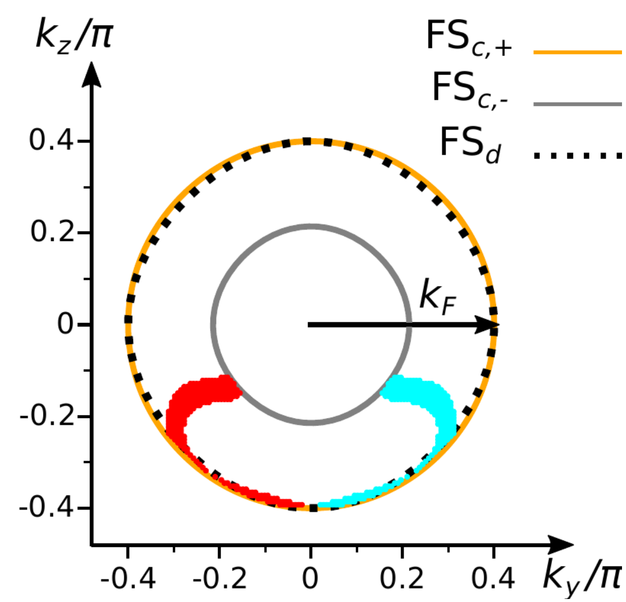


Basis: $\left(\chi_+(\mathbf{k}), d^\dagger(-\mathbf{k})\right)$

- Nontrivial example of a general statement that Nielsen-Ninomiya theorem does not hold in lattice systems when $\mathrm{U}(1)$ symmetry is broken

Surface modes:

- Nontrivial pairing phase winding near gap node exhibits *local* chiral p-wave symmetry and leads to zero energy surface modes
- Chiral surface modes originate from BdG Weyl node and merge into unpaired Fermi surface



Fractional Mermin-Ho relation

Spinor pairing order parameter: $\Delta(\mathbf{r}) = |\Delta(\mathbf{r})| e^{i\phi(\mathbf{r})} \eta(\mathbf{r})$ $\quad$ $\eta(\mathbf{r}) = \begin{pmatrix} \eta_\uparrow(\mathbf{r}) \\ \eta_\downarrow(\mathbf{r}) \end{pmatrix}, \eta^\dagger \eta = 1$

Superfluid velocity:

$$\mathbf{v}_s = -i \frac{\hbar}{m^*} \frac{1}{2} \left(\Delta^\dagger \boldsymbol{\nabla} \Delta - \mathrm{h.c.} \right)$$

$$= \frac{\hbar}{m^*} \Big(\underbrace{\boldsymbol{\nabla}\phi(r)}_{\text{Contribution from U(1) phase}} - \underbrace{i\, \eta_\alpha^*(\mathbf{r}) \boldsymbol{\nabla} \eta_\alpha(\mathbf{r})}_{\text{Contribution from spatial texture of } \eta} \Big) \qquad (\alpha = \uparrow, \downarrow)$$

Curl of superfluid velocity:

$$(\boldsymbol{\nabla} \times \mathbf{v}_s)_i = -i \frac{\hbar}{m^*} \epsilon_{ijk} \partial_j \eta_\alpha^* \partial_k \eta_\alpha$$

(No singularities in $\phi(\mathbf{r})$)

Hopf map ($S^3 \to S^2$): $\hat{n}^i = \eta^\dagger \sigma^i \eta$

Fractional Mermin-Ho relation:

$$(\boldsymbol{\nabla} \times \mathbf{v}_s)_i = \frac{1}{4} \frac{\hbar}{m^*} \epsilon_{abc} \epsilon_{ijk} \hat{n}^a \partial_j \hat{n}^b \partial_k \hat{n}^c$$

$\Rightarrow$ Nontrivial spatial texture of $\hat{n}(\mathbf{r})$ can lead to coreless vortices

^{3}He-A Phase	Spinor superconductor
$\Delta_{\mu j}(\mathbf{r}) = \lvert\Delta_0(\mathbf{r})\rvert e^{i\phi(\mathbf{r})}\hat{d}(\mathbf{r})\left(\hat{m}_j(\mathbf{r}) + i\hat{n}_j(\mathbf{r})\right)$	$\Delta(\mathbf{r}) = \lvert\Delta(\mathbf{r})\rvert e^{i\phi(\mathbf{r})}\eta(\mathbf{r}), \quad \eta(\mathbf{r}) = \begin{pmatrix}\eta_\uparrow(\mathbf{r}) \\ \eta_\downarrow(\mathbf{r})\end{pmatrix}$
$\mathbf{v}_s = \frac{\hbar}{m^*}\left(\nabla\phi(r) - i\,\underbrace{\hat{\mathbf{m}}(\mathbf{r})\cdot\nabla\hat{\mathbf{n}}(\mathbf{r})}_{\text{Contribution from orbital texture}}\right)$	$\mathbf{v}_s = \frac{\hbar}{m^*}\left(\nabla\phi(r) - i\,\underbrace{\eta_\alpha^*(\mathbf{r})\nabla\eta_\alpha(\mathbf{r})}_{\text{Contribution from spinor texture}}\right)$
Orbital angular momentum: $\hat{\mathbf{l}}(\mathbf{r}) = \hat{\mathbf{m}}(\mathbf{r}) \times \hat{\mathbf{n}}(\mathbf{r})$	**Pseudospin:** $\hat{\mathbf{n}}(\mathbf{r}) = \eta^\dagger(\mathbf{r})\boldsymbol{\sigma}\,\eta(\mathbf{r})$
Mermin-Ho Relation: $(\nabla \times \mathbf{v}_s)_i = \frac{1}{2}\frac{\hbar}{m^*}\epsilon_{abc}\epsilon_{ijk}\hat{l}^a\partial_j\hat{l}^b\partial_k\hat{l}^c$ Mermin and Ho, PRL **36**, 594 (1976)	***Fractional* Mermin-Ho Relation:** $(\nabla \times \mathbf{v}_s)_i = \frac{1}{4}\frac{\hbar}{m^*}\epsilon_{abc}\epsilon_{ijk}\hat{n}^a\partial_j\hat{n}^b\partial_k\hat{n}^c$
Superfluid can acquire nontrivial circulation from spatial inhomogeneity of **the orbital texture**	Superfluid can acquire nontrivial circulation from spatial inhomogeneity of **the pseudospin texture**

Conclusion

- We have introduced a model for which pairing occurs between a topologically trivial and nontrivial Fermi surface.

- The half-integer pair monopole charge leads to pairing order with **half-integer partial wave symmetry following singular spinor representation**, even though it arises from Cooper pairs which are bosonic.

- There is a ***single*** **emergent BdG Weyl node**, demonstrating a “violation” of Nielsen Ninomiya theorem.

- Superfluid velocity obeys a **fractional Mermin-Ho relation** for a spatially inhomogeneous spinor pairing order.

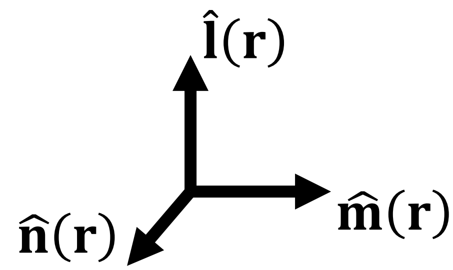
arXiv: 2409.09579

Extra Slides

Superfluid ^{3}He-A and Mermin-Ho Relation

Pairing order parameter of ^{3}He-A:

$$\hat{\Delta}_{\mu j}(\mathbf{r}) = \underbrace{e^{i\phi(\mathbf{r})}}_{\text{U(1) phase}} \times \underbrace{\hat{d}_{\mu}(\mathbf{r})}_{\text{Spin}} \times \left(\underbrace{m_j(\mathbf{r}) + i n_j(\mathbf{r})}_{\text{Orbital}}\right)$$



Orbital angular momentum: $\hat{\mathbf{l}}(\mathbf{r}) = \hat{\mathbf{n}}(\mathbf{r}) \times \hat{\mathbf{m}}(\mathbf{r})$

Superfluid velocity:

$$v_{s,j} = \frac{\hbar}{m^*}\left(\underbrace{\partial_j \phi(\mathbf{r})}_{\text{Contribution from U(1) phase}} + \underbrace{\hat{\mathbf{m}}(\mathbf{r}) \cdot \partial_j \hat{\mathbf{n}}(\mathbf{r})}_{\text{Contribution from orbital texture}}\right)$$

Mermin-Ho Relation:

$$(\nabla \times \mathbf{v}_s)_i = \frac{1}{2}\frac{\hbar}{m^*}\epsilon^{abc}\epsilon_{ijk}\hat{l}^a \partial_j \hat{l}^b \partial_k \hat{l}^c$$

Superfluid can acquire nontrivial circulation from spatial inhomogeneity of the l-vector

Mermin and Ho, PRL **36**, 594 (1976)